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FRATTINI CLOSED GROUPS AND ADEQUATE EXTENSIONS OF GLOBAL FIELDS

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ABSTRACT

Let L be a finite Galois extension of a global field F. It is shown that if the Galois group $G = \mathrm{Gal}(L/F)$ satisfies a certain condition, then L is a maximal commutative subfield of some F-division algebra if and only if the intermediate field corresponding to the Frattini subgroup of G is also a maximal commutative subfield of some F-division algebra. In particular this condition holds if G is a supersolvable group.

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1. Introduction

This paper is concerned with the following question: If F is a field and L a finite extension of F, does there exist an F-central division algebra D containing L as a maximal commutative subfield? If there exists such a D, L is said to be F-adequate; otherwise L is F-deficient. This question was first explored in depth in [S].

In this paper we consider a certain group-theoretic condition and show that if a finite Galois extension L of a global field F has a Galois group G satisfying this condition, then the F-adequacy of L is determined by the F-adequacy of the subfield fixed by the Frattini subgroup of G. We show in Section 3 that a class of groups that includes finite supersolvable groups satisfies this condition. In Section 4 we exhibit some groups that do not satisfiy this condition. For basic group-theoretic results and facts about Frattini subgroups in particular, the reader is referred to [D] and [H].

We fix L to be a finite Galois extension of a global field F with $G = \mathrm{Gal}(L/F)$. We write $\Phi(G)$ for the Frattini subgroup of G, and let K be the subfield of L that is fixed by $\Phi(G)$. Since $\Phi(G)$ is normal in G, it follows that K/F is a Galois extension.

A result of Schacher ([S; Corollary 2.3]) implies that if L is F-adequate, then K is also F-adequate. We show in Theorem 2.2 that if G satisfies the group-theoretic condition mentioned above, then the converse of this statement holds.

Combining Propositions 2.1, 2.5 and 2.6 of [S], we have the following characterization of F-adequate Galois extensions. This characterization will allow us to connect the F-adequacy of L to the F-adequacy of K.

THEOREM 1.1: *[S] Let L/F be a Galois extension of global fields with $[L : F] = p_1^{e_1} \cdots p_r^{e_r}$, where $p_1, \ldots, p_r$ are distinct prime numbers. The following are equivalent.*

(1) *L is F-adequate.*
(2) *For each i, $1 \le i \le r$, there exist two distinct prime spots q, q' of F (depending on i) such that $p_i^{e_i} \mid [L_q : F_q]$ and $p_i^{e_i} \mid [L_{q'} : F_{q'}]$.*

2. A group-theoretic condition for adequacy

Definition 2.1: Let p be a prime number. We say a finite group G is **p-Frattini closed** if for every subgroup H of G, if $p \mid [G : H]$, then $p \mid [G : \Phi(G)H]$. We say G is **Frattini closed** if G is p-Frattini closed for all primes p.

We remark that it is always true that if $[G : H] > 1$, then $[G : \Phi(G)H] > 1$. For if $[G : H] > 1$, then H is contained in a maximal subgroup M of G. Then $\Phi(G) \leq M$ and so $\Phi(G)H \leq M \lneq G$.

THEOREM 2.2: *Let L be a finite Galois extension of a global field F with $G = \mathrm{Gal}(L/F)$. Let $\Phi(G)$ be the Frattini subgroup of G, and let K be the subfield of L that is fixed by $\Phi(G)$. If K is F-adequate and G is Frattini closed, then L is F-adequate.*

Proof: Let $[L : F] = p_1^{e_1} \cdots p_r^{e_r}$ and $[K : F] = p_1^{f_1} \cdots p_r^{f_r}$, where $f_i \leq e_i$, $1 \leq i \leq r$. Fix p_i. Since K is F-adequate, there exist distinct prime spots q, q' of F such that $p_i^{f_i} \mid [K_q : F_q]$ and $p_i^{f_i} \mid [K_{q'} : F_{q'}]$. Let $H = \mathrm{Gal}(L/(F_q \cap L))$. Since $L_q = F_q L, K_q = F_q K$, and both $L/F, K/F$ are Galois extensions, we have $\mathrm{Gal}(L_q/F_q) \cong \mathrm{Gal}(L/(F_q \cap L))$ and $\mathrm{Gal}(K_q/F_q) \cong \mathrm{Gal}(K/(F_q \cap K))$. Since $\Phi(G)$ corresponds to K and H corresponds to $F_q \cap L$, it follows that the group $\Phi(G)H$ corresponds to $K \cap (F_q \cap L) = F_q \cap K$. We have $p_i^{f_i} \mid [K_q : F_q] = [K : F_q \cap K] = [\Phi(G)H : \Phi(G)]$, and $[G : \Phi(G)] = [K : F] = p_1^{f_1} \cdots p_r^{f_r}$. Therefore $p_i \nmid \frac{[G:\Phi(G)]}{[\Phi(G)H:\Phi(G)]} = [G : \Phi(G)H]$. Thus $p_i \nmid [G : H] = \frac{p_i^{e_1} \cdots p_r^{e_r}}{|H|}$, since G is Frattini closed. Therefore $p_i^{e_i} \mid |H| = [L : F_q \cap L] = [L_q : F_q]$. Since a similar calculation holds for q', it follows from Theorem 1.1 that L is F-adequate. ∎

The usefulness of this proposition depends on how $\mathrm{Gal}(K/F) = G/\Phi(G)$ compares to G. For example, if $\Phi(G) = 1$, then G is trivially Frattini closed and $K{=}L$, so Theorem 2.2 provides no new information. This holds, for example, if G is simple, or $G = S_n, n \geq 2$, or $G = A_4$. On the other hand, if $\Phi(G)$ is large, then Theorem 2.2 reduces the question of F-adequacy of L to that of a much smaller field K. The next result shows that p-groups and nilpotent groups are always Frattini closed.

PROPOSITION 2.3: *Let G be a finite group. Then G is Frattini closed if either G is a p-group or G is a nilpotent group.*

Proof: Assume first that G is a p-group, and suppose $p \mid [G : H]$. Then $[G : H] > 1$, and so $[G : \Phi(G)H] > 1$ by our earlier remark. Then $p \mid [G : \Phi(G)H]$ since G is a p-group.

Now assume G is nilpotent. Then $G \cong P_1 \times \cdots \times P_r$ where $P_1, \ldots, P_r$ are the Sylow subgroups of G. Since the Sylow subgroups have pairwise relatively prime orders, it follows that for any subgroup H of G, $H \cong H_1 \times \cdots \times H_r, H_i \leq P_i$.

Also $\Phi(G) = \Phi(P_1) \times \cdots \times \Phi(P_r)$. Then $[G : \Phi(G)H] = \prod_{i=1}^{r}[P_i : \Phi(P_i)H_i]$ and $[G : H] = \prod_{i=1}^{r}[P_i : H_i]$. The result then follows from the result for p-groups. ∎

Recall that if G is a p-group, then $G/\Phi(G)$ is an elementary abelian p-group of rank equal to the order of a minimal set of generators for G, and if G is a nilpotent group, then $G/\Phi(G)$ is a direct product of elementary abelian p-groups for various primes p. The results above show that questions of adequacy for Galois p-extensions of global fields reduce to questions of adequacy for elementary abelian p-extensions. In fact, the following two results show that this is true for Galois extensions having nilpotent Galois groups.

PROPOSITION 2.4: *Let E_1/F and E_2/F be Galois extensions of a global field F with $[E_1 : F]$ and $[E_2 : F]$ relatively prime. Then the compositum $E = E_1E_2$ is F-adequate if and only if E_1 and E_2 are both F-adequate.*

Proof: First observe that by [S; Corollary 2.3], if E is F-adequate, then E_1 and E_2 are both F-adequate. Conversely, suppose E_1 and E_2 are both F-adequate. Let p be a rational prime. Suppose $p^n \mid [E : F]$ and $p^{n+1} \nmid [E : F]$. Then $p^n \mid [E_i : F]$ for $i = 1$ or $i = 2$. The F-adequacy of E_i implies the existence of two primes q, q' of F such that $p^n \mid [(E_i)_q : F_q]$ and $p^n \mid [(E_i)_{q'} : F_{q'}]$, and hence $p^n \mid [E_q : F_q]$ and $p^n \mid [E_{q'} : F_{q'}]$. Thus E is F-adequate by Theorem 1.1. ∎

COROLLARY 2.5: *Let L be a finite Galois extension of a global field F with nilpotent Galois group G. Then L is F-adequate if and only if E is F-adequate for every maximal elementary abelian p-extension E of F inside L.*

Proof: Since G is Frattini closed by Proposition 2.3, it follows from Theorem 2.2 that L is F-adequate if and only if K is F-adequate, where $\mathrm{Gal}(K/F) = G/\Phi(G)$. Since $G/\Phi(G)$ is a direct product of the maximal elementary p-abelian quotients of G, we see that K is the compositum of the maximal elementary abelian p-extensions of F inside L. Then by Proposition 2.4, it follows that K is F-adequate if and only if each of these maximal elementary abelian p-extensions is F-adequate. ∎

For a p-group (or nilpotent group) with a small number of generators relative to its order, Theorem 2.2 gives a significant reduction. This becomes particularly useful when combined with the following results of Schacher.

PROPOSITION 2.6:

(1) *([S; Theorem 10.1]) Let $[k : \mathbb{Q}] = n$ and let G be a finite p-group. If there exists a k-adequate Galois extension of k with Galois group G, then the number of generators of G is at most $(n/2) + 2$.*

(2) *([S; Theorem 10.2]) Let k be an algebraic number field in which p has a unique extension, and let G be a finite group. If there exists a k-adequate Galois extension of k with Galois group G, then any p-Sylow subgroup of G is metacyclic.*

(3) *([S; Theorem 10.3]) Let k be a global field of characteristic $p \neq 0$ and let G be a finite group. If there exists a k-adequate Galois extension of k with Galois group G, then every q-Sylow subgroup is metacyclic, for $q \neq p$.*

Statement (1) of Proposition 2.6 shows that if L is a $\mathbb{Q}$-adequate Galois p-extension of $\mathbb{Q}$, then $G/\Phi(G) \cong \mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Thus by Theorem 2.2 and Proposition 2.3, a Galois p-extension L is $\mathbb{Q}$-adequate if and only if the maximal elementary abelian p-extension of $\mathbb{Q}$ inside L has degree at most p^2, and that extension is $\mathbb{Q}$-adequate. This result is used in [GLS] to give an explicit means for determining $\mathbb{Q}$-adequacy of all Galois 2-extensions of $\mathbb{Q}$.

3. Frattini-closed groups

This section considers which finite groups are Frattini closed. For example, we show in Corollary 3.8 that finite supersolvable groups are always Frattini closed.

PROPOSITION 3.1: *Let P be a Sylow p-subgroup of a finite group G. If G is p-Frattini closed, then $\Phi(G) \cap P \leq \Phi(P)$.*

Proof: Let M be a maximal subgroup of P. Then $[P : M] = p$, so $p \mid [G : M]$. Since G is p-Frattini closed, it follows that $p \mid [G : \Phi(G)M]$, and hence $P \not\leq \Phi(G)M$. This implies $(\Phi(G) \cap P)M \lneq P$. Therefore $\Phi(G) \cap P \leq M$, since M is a maximal subgroup of P. This holds for all such M, and so $\Phi(G) \cap P \leq \Phi(P)$. ∎

The following lemma is proved in [I; Lemma 8.22], pp. 108–109.

LEMMA 3.2: *Let N be a normal subgroup of a finite group G, H any subgroup of G, and p a prime number. Suppose that P is a Sylow p-subgroup of G such that $P \cap H$ is a Sylow p-subgroup of H. Then $P \cap NH = (P \cap N)(P \cap H)$ is a Sylow p-subgroup of NH.*

We observe that if $\Phi(G) \cap P \leq \Phi(P)$ for some Sylow p-subgroup P of G, then this condition holds for every Sylow p-subgroup of G, since $\Phi(G)$ is normal in G.

PROPOSITION 3.3: *Let G be a finite group. If $\Phi(G) \cap P \leq \Phi(P)$ for some (and hence every) Sylow p-subgroup P of G, then G is p-Frattini closed.*

Proof: Let H be a subgroup of G, and suppose $p \mid [G : H]$. Let J be a Sylow p-subgroup of H and let P be a Sylow p-subgroup of G that contains J. Then $P \cap H = J$. Let $K = \Phi(G) \cap P \leq \Phi(P)$. Then K is a Sylow p-subgroup of $\Phi(G)$ since $\Phi(G)$ is normal in G. We know $p \mid [P : J]$, since $p \mid [G : H]$ implies $p \mid [G : J] = [G : P][P : J]$, but $p \nmid [G : P]$. Then $p \mid [P : \Phi(P)J]$ since P is p-Frattini closed by Proposition 2.3. Therefore $p \mid [P : KJ]$, since $K \leq \Phi(P)$, and hence $p \mid [G : KJ]$. Lemma 3.2 then implies $p \mid [G : \Phi(G)H]$ since KJ is a Sylow p-subgroup of $\Phi(G)H$. Thus G is p-Frattini closed. ∎

COROLLARY 3.4: *Let G be a finite group, and let P be a Sylow p-subgroup of G. Then G is p-Frattini closed if and only if $\Phi(G) \cap P \leq \Phi(P)$.*

Proof: This follows immediately from Propositions 3.1 and 3.3. ∎

LEMMA 3.5: *Let G be a finite group and let P be a Sylow p-subgroup of G. Let N be a normal subgroup of G, and assume that either p does not divide $|N|$ or $N \leq \Phi(P)$. If G/N is p-Frattini closed, then G is p-Frattini closed.*

Proof: First suppose p does not divide $|N|$. Let H be a subgroup of G with $p \mid [G : H]$. Then $p \mid [G : NH]$, since $[G : H] = [G : NH][NH : H] = [G : NH][N : N \cap H]$ and $p \nmid [N : N \cap H]$. Since $p \mid [G : NH] = [G/N : NH/N]$ and G/N is p-Frattini closed, it follows that $p \mid [G/N : \Phi(G/N)NH/N]$. Since $\Phi(G/N) \geq \Phi(G)N/N$, it follows that

$$p \mid [G/N : (\Phi(G)N/N)(NH/N)] = [G/N : \Phi(G)HN/N] = [G : \Phi(G)HN].$$

Therefore $p \mid [G : \Phi(G)H]$, and G is p-Frattini closed.

Now suppose $N \leq \Phi(P) \leq P$. We know P/N is a Sylow p-subgroup of G/N since $[G/N : P/N] = [G : P]$. Since G/N is p-Frattini closed, we have, by Proposition 3.1,

$$(\Phi(G)N/N) \cap (P/N) \leq \Phi(G/N) \cap P/N \leq \Phi(P/N) = \Phi(P)/N.$$

Thus $\Phi(G)N \cap P \leq \Phi(P)$. Therefore $\Phi(G) \cap P \leq \Phi(P)$, and G is p-Frattini closed by Proposition 3.3. ∎

If G is a finite group and p a prime number, let $O_{p'}(G)$ denote the largest normal subgroup of G whose order is prime to p.

Definition 3.6: Let G be a finite group and p a prime number. We say that G has **p-length** 1 if $G/O_{p'}(G)$ has a normal Sylow p-subgroup.

We note that if G has p-length 1 and N is a normal subgroup of G, then G/N also has p-length 1.

THEOREM 3.7: *Let G be a finite group of p-length 1. Then G is p-Frattini closed.*

Proof: The proof is by induction on $|G|$. Let P be a Sylow p-subgroup of G. If $|O_{p'}(G)| > 1$, then $G/O_{p'}(G)$ has p-length 1 and by induction it follows that $G/O_{p'}(G)$ is p-Frattini closed. Then Lemma 3.5 implies that G is p-Frattini closed. We may now assume that $|O_{p'}(G)| = 1$. Then P is a normal subgroup of G and it follows that $\Phi(P)$ is also a normal subgroup of G. If $|\Phi(P)| > 1$, then $G/\Phi(P)$ has p-length 1 and by induction $G/\Phi(P)$ is p-Frattini closed. Lemma 3.5 again implies that G is p-Frattini closed. We may now assume that $|\Phi(P)| = 1$ so that P is a normal elementary abelian Sylow p-subgroup of G. By the Schur–Zassenhaus Theorem, there exists a p-complement K to P in G. We may regard P as an $F_p[K]$-module. By Maschke's Theorem, $P = P_1 \oplus \cdots \oplus P_r$ with each P_i an irreducible $F_p[K]$-module. Let $P^i = \Sigma_{j \neq i} P_j$. Then $P^i K$ is a maximal subgroup of G and so $\Phi(G) \leq \bigcap_i P^i K = K$. Hence $\Phi(G) \cap P = 1$, as desired. ∎

COROLLARY 3.8: *Let G be a finite supersolvable group. Then G has p-length 1 for every prime p and hence G is Frattini closed.*

Proof: Let p be a prime divisor of $|G|$ and let P be a Sylow p-subgroup of G. By [H; 10.5.4], $[G,G]$ is nilpotent. So $[G,G] = (P \cap [G,G]) \times R$, where R is a normal subgroup of G of order prime to p. Hence $R \leq O_{p'}(G)$ and so, $[G,G] \leq O_{p'}(G)P$. Thus $O_{p'}(G)P$ is a normal subgroup of G. Therefore G has p-length 1 and so, by Theorem 3.7, G is p-Frattini closed. ∎

4. Some counterexamples

In this section we describe a few examples of finite groups which are not Frattini closed. By Frattini's argument ([H; 10.4.2]), $\Phi(G)$ is a normal, nilpotent subgroup of the finite group G, hence contained in the Fitting subgroup $F(G)$, the maximal normal nilpotent subgroup of G. In all of our examples, $F(G)$ will be an elementary abelian normal p-subgroup of G for some prime p and we will have $C_G(F(G)) = F(G)$. Hence we may regard $F(G) = V$ as a $K[H]$-module, where $K = \mathbb{Z}/p\mathbb{Z}$ and $H = G/F(G)$. We denote by $J(V)$ the intersection of all maximal $K[H]$-submodules of V.

LEMMA 4.1: *For G and V as above, $J(V) \leq \Phi(G) \leq V$.*

Proof: As noted above, $\Phi(G) \leq V$. Let M be a maximal subgroup of G. If $G \neq VM$, then $V \leq M$. On the other hand, if $G = VM$, then $V \cap M < V$ and so $V \cap M$ is contained in a maximal $K[H]$-submodule W of V, whence $WM < G$ and so $W = V \cap M$. Thus either $\Phi(G) = V$ or $\Phi(G)$ is the intersection of certain maximal $K[H]$-submodules of V. In particular $J(V) \leq \Phi(G)$, as claimed. ∎

In most of our examples, $G = V \rtimes H$ for H a subgroup of G having a Sylow p-subgroup of order p.

LEMMA 4.2: *Suppose that $G = V \rtimes H$ where H is a subgroup of G having a Sylow p-subgroup A of order p. Let $P = VA$. Then $\Phi(P) = [V, A] = [P, P]$.*

Proof: Since V is an elementary abelian p-subgroup of G and $[V, A] \triangleleft VA = P$, it follows that $P/[V, A]$ is an abelian group generated by elements of order p, hence is elementary abelian. Thus $\Phi(P) \leq [V, A] \leq [P, P]$, whence equality holds. ∎

Since our goal is to exhibit examples with $\Phi(G) \cap P \not\leq \Phi(P)$ for P a Sylow p-subgroup of G, it will suffice to exhibit groups $G = V \rtimes H$ having a Sylow p-subgroup $P = V \rtimes A$ with $|A| = p$ such that $J(V) \not\leq [V, A]$. If V is completely reducible, then $J(V) = 0$ and so $J(V) \leq [V, A]$. Hence we will look at the opposite extreme — indecomposable $K[G]$-modules which are not irreducible. We shall look at projective indecomposable modules for groups H with $H = LA$ where A is a Sylow p-subgroup of H of order p and L is a Hall p'-subgroup of H, i.e., $A \cap L = 1$. Such Hall subgroups always exist in solvable groups by a theorem of Philip Hall. The following observation was called to our attention by Walter Feit.

LEMMA 4.3: *Let H be a finite group, p a prime and K a field of characteristic p. Suppose that H has a Hall p'-subgroup L. Then every projective indecomposable $K[H]$-module is a direct summand of $\mathrm{Ind}_L^H(W)$ for some irreducible $K[L]$-module.*

Proof: Projective indecomposables are direct summands of the group algebra $K[H]$. Now $K[H] = \mathrm{Ind}_1^H(K)$ where 1 denotes the trivial subgroup of H. Then by transitivity of induction

$$K[H] = \mathrm{Ind}_L^H(\mathrm{Ind}_1^L(K)) = \mathrm{Ind}_L^H(K[L]).$$

As L is a p'-group, $K[L]$ is completely reducible by Maschke's Theorem. Hence $\mathrm{Ind}_L^H(K[L])$ is a sum (with multiplicities) of $\mathrm{Ind}_L^H(W)$ as W ranges over the irreducible $K[L]$-modules. Now we are done by the Krull–Schmidt Theorem. ∎

In our first example, $p = 3$, $K = \mathbb{Z}/3\mathbb{Z}$ and $H = \mathrm{SL}(2,3) = Q \rtimes A$, where Q is a normal quaternion subgroup of order 8 and A is a cyclic group of order 3. Let W be the absolutely irreducible 2-dimensional $K[Q]$-module obtained by regarding Q as a subgroup of $\mathrm{GL}(2,3)$. By Green's Indecomposability Theorem [CR; 19.22], $V = \mathrm{Ind}_Q^H(W)$ is a projective indecomposable $K[H]$-module of dimension 6. Such projective indecomposable modules always have a unique irreducible quotient. As W is the restriction to Q of an irreducible 2-dimensional $K[H]$-module V_0, it follows by Frobenius–Nakayama Reciprocity [A; III.6] that $J(V)$ is 4-dimensional with $V/J(V) \cong V_0$. On the other hand, as a $K[A]$-module, V is projective, hence free. So V_A is the direct sum of two copies of $K[A]$, whence $[V, A]$ is also 4-dimensional.

LEMMA 4.4: *Let $H = \mathrm{SL}(2,3)$, $K = \mathbb{Z}/3\mathbb{Z}$ and let V be the 6-dimensional indecomposable $K[H]$-module described above. Let A be a Sylow 3-subgroup of H. Then $J(V) \not\leq [V, A]$. Hence if $G = V \rtimes H$, then $\Phi(G) \cap P \not\leq \Phi(P)$.*

Proof: Suppose on the contrary that $J(V) \leq [V, A]$. Then as both spaces are 4-dimensional, $J(V) = [V, A]$ and so A is in the kernel of the action of H on $V/J(V)$. However $V/J(V) \cong V_0$, a faithful $K[H]$-module. This contradiction proves that $J(V) \not\leq [V, A]$ and the final statement follows by previous remarks. ∎

Remark: With H and V as above, V has a unique minimal $K[H]$-submodule V_0 of dimension 2. If we set $V_1 = V/V_0$, then H and V_1 afford an even smaller counterexample.

We shall only remark briefly on other similar examples. First of all, we may mimic the above example by replacing $\mathrm{SL}(2,3)$ by $H = Q \rtimes A$, where Q is a non-abelian q-group for some prime $q \neq p$ and A is a cyclic group of order p, where p divides $q^2 - 1$ and $C_H(Q) = Z(Q)$. We let K be a finite field of characteristic p containing primitive q^{th} roots of 1. Then there is an absolutely irreducible $K[Q]$-module W of dimension q and again $V = \mathrm{Ind}_Q^H(W)$ is an indecomposable $K[H]$-module with $V/J(V) \cong W_0$, where W_0 is an irreducible lift of W to a q-dimensional $K[H]$-module. Again V_A is a free $K[A]$-module, whence

$\dim_K([V,A]) = \dim_K(J(V))$. Thus the hypothesis that $J(V) \leq [V,A]$ leads to the same contradiction as before.

Finally we mention two non-solvable counterexamples. First, taking $p = 5$ and $H = A_5$, the alternating group of degree 5, we see that H has a Hall $5'$-subgroup J isomorphic to A_4. Taking $K = \mathbb{Z}/5\mathbb{Z}$, we may take an irreducible $K[J]$-module W of dimension 3 and then take $V = \mathrm{Ind}_J^H(W)$. Although Green's Theorem does not apply, an easy inspection of $K[H]$ confirms that V is a projective indecomposable module for H, after which the same argument as before shows that $J(V) \not\leq [V,A]$ for A a Sylow 5-subgroup of H.

Lastly, a rather different type of counterexample is afforded by a group G with $F(G) = V$ elementary abelian of order 8 and with $G/V \cong \mathrm{GL}(3,2)$ but with no subgroup of G isomorphic to $\mathrm{GL}(3,2)$. Such a group may be exhibited as a subgroup of the Chevalley group $G_2(5)$, for example. Since there is no partial complement to V in G, it follows easily that $\Phi(G) = V$. On the other hand, G contains the normalizer of a Cartan subgroup T of $G_2(5)$ with

$$T \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$$

and with $N_{G_2(5)}(T)/T \cong W(G_2) \cong D_6$, where D_6 is the dihedral group of all symmetries of the regular hexagon. A Sylow 2-subgroup P of G is contained in $N_{G_2(5)}(T)$, whence by inspection, $\Phi(P)$ is contained in T. Hence $\Phi(G) \cap P = \Phi(G) = V \not\leq \Phi(P)$ in this case as well.

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